Solution of Systems of Linear Equations

by the Use of Punched Card Equipment

1. Introduction

A set of equations such as

$2x + 3y = 8$ $6x - y = 4$	2 equations 2 unknowns

or

$3x - 4y + 5z = 16$ $2x - 3y + z = 5$ $-4x + y - z = 29$	3 equations 3 unknowns

or sets containing more equations and more unknowns are called systems of linear equations. The solution of such a system of equations consists in finding a set of values, one for each of the unknown variables, that will cause each equation to be satisfied. In order that there be one and only one such solution (or set of values) for the problem it is usual to require that the number of unknowns and the number of equations be equal. We shall tacitly assume that our systems of equations are of this kind altho the method we shall give can be applied to find the solution of any system insofar as such solutions exist.

The solution of such linear systems occurs as part of the solution of many practical problems. Each correlation requires such a solution, and many other applications are found in statistics. Electrical circuit analysis and the analysis of structures are two engineering fields requiring the frequent solution of large linear systems of equations.

The exact or approximate solution of linear ordinary and partial differential equations can be reduced to the solution of a linear system of algebraic equations. Because of the rapidly growing demand for approximate solutions of differential equations in the physical sciences and in engineering, it is likely that this application of any rapid method of solving large systems of equations would be most important.

If the system of equations contains only a few unknowns and equations, the solution can be conveniently carried out with no mechanical aid or by the use of a slide rule or calculating machine. However, if $\underline{n}$ is the number of equations and unknowns, the work of solving the system increases approximately as fast as $\underline{n}^3$. So if four equations and four unknowns require twenty minutes for solution (this time of course depends on the accuracy required and the skill of the computor) sixteen equations and sixteen unknowns would require twenty-one hours of work. In solving some systems of equations the work can be arranged for a check at each stage of the solution but in other cases there is no check until the end and the chance of error is large. An error made at the beginning of the calculation naturally invalidates all work beyond that point. Hence, the practical solution of even a fairly large system of equations is a laborious process. Any practical method of decreasing this labor will be of great value in the various applications.

The mechanical method which will be described required the use of the standard tabulator with summary punch of the

International Business Machines Corporation. Only slight modifications of the tabulator will be required. To adapt this equipment to the present problem one piece of auxilliary apparatus is needed which will be plugged into the tabulator when it is used for solving equations. The coefficients of the equations and the constant term are punched on cards and the entire process (except one final division) is carried thru without any opportunity for the operator to introduce errors, except thru gross mismanagement of the process. Such gross errors would be immediately evident.

2. Mathematics of Solution

Before discussing the punched card method of solving equations it is essential to have well in mind the mathematical details of such solutions. The most elegant method of studying the soluti ns of linear systems of equations makes use of determinants and these quantities can be used in discussing the mechanical method. On the contrary, we shall use the method of elimination of variables since it makes the mechanical method equally clear and is simpler. However, it should be remembered that the connection between solution of systems of equations and evaluation of determinants is very close and the method given is readily adaptable to the latter purpose.

If we have two linear equations in any number of unknown quantities, they may be combined in such a way as to eliminate one variable and leave one equation containing one less unknown. For example consider the equations:

$$123.5x + 251.4y - 378.5z = 4982\ ,$$
$$50.7x - 361.2y + 488.1z = 1174\ .$$

If one multiplies the second equation by $\frac{123.5}{50.7} = 2.436$ and subtracts it from the first there results

$$1131.3y - 1567.5z = 2122.$$

This equation is satisfied by the same values of _y_ and _z_ that will simultaneously satisfy the two former equations but the _x_ is missing.

If we start out with nine equations in nine unknowns we may select eight pairs of equations in which each equation appears at least once. If we eliminate some one unknown from each pair we will have eight equations in eight unknowns. By repeating this process we will obtain in turn seven equations in seven unknowns etc., and finally one equation in one unknown. The value of this unknown is at once found by a division. The values of the other unknowns are found by substitution. Substituting in one of the two equations in two unknowns will yield a value for a second unknown; then by substituting in one of the three equations in three unknowns, a value for a third unknown may be found and etc.. In this way the values of each of the unknowns which taken together will simultaneously satisfy each equation can be found.

4. _Mechanical Method_

In the mechanical method here described the coefficients and constant terms are punched on cards, one equation on each card. Negative values are indicated by punching complements.

When two such cards are properly presented the tabulator computes the new equation resulting from the elimination of one variable between the two original equations, and the summary punch punches a card representing this new equation. This process may be repeated until only one equation in one unknown remains. The value of one of the unknowns is thus obtained. The substitutions necessary to obtain the other unknowns are readily carried out on the machines since, for example, the substitution of $x = 5$ in $2x + 3y = 16$ is equivalent to the elimination of x between $2x + 3y = 16$ and $x + 0.y = 5$. It is therefore clear that the fundamental element of the machine solution of systems of equations is the process by which the tabulator eliminates one variable between two equations. The details of this process will now be given. We shall first describe an arrangement for this purpose in which the control of the process is in the hands of the operator. The more complicated elaborations necessary to make the process more nearly automatic will be added later.

In this method cards are not presented at the brushes in the usual way since the card would then pass out of the machine and in this application it is essential that the cards be held. Instead the card is placed in the auxilliary apparatus in which it rests in the proper position upon a set of ten horizontal bars indicated by G, G', G" etc. in the schematic drawing. Above the card are rows of brush contacts indicated by H, H', H", etc. At the holes punched in the card these brushes make contact with the bars below. These bars receive current from the master breaker thru a contral box E and a distributor D which is mounted on the timing shaft of the tabulator. This distribution is so timed that when the shaft is in the "0" position the corresponding impulse will pass into bar G, the "1" impulse will pass into bar G' etc. It is obvious that if the brush contacts H, H' etc. were plugged directly into a counter, at each cycle of the machine the card contents would add into the counter. Some arrangement will be necessary to block out the action of the card switch which stops the machine and interrupts the circuit when no cards are passing the machine in the usual way.

However the brush contacts H, H', etc. are not plugged directly into the counters but are connected to contact springs C, C', etc. fastened to a sliding bar of insulating material K. These contact springs touch contact points P,P' etc. connected to the jacks that may be plugged into the various counters of the tabulator; when however the bar K is shifted the relation between the fields on the card and the counters is changed in much the same way and for exactly the same purpose that the carriage on a computing machine is shifted, The schematic drawing is drawn on the basis of fields of eight places but this is of course optional.

As has been stated negative numbers are punched on the cards as complements. When the bar K is shifted to the right the contact points P, P' etc. left unconnected with H, H', etc.

must be supplied with a "9" impulse if the number punched in the corresponding field is a complement. To supply this impulse the additional groups B,B', etc. of contact springs are provided. The "9" impulse is drawn from the "9" point of the distributor thru manual switches closed if the number in the associated field is a complement, or better yet from the left brush contact of any field such as H thru switches operated by the timing shaft and closed only at the "9" position. In this case a "9" punched in the left column of any field indicates a complement but this only slightly decreases the capacity of the machine since any other number punched in this column will not interfere with the proper operation of the machine and since in any case care will have to be exercised to avoid a carry over being lost.

The control E has two keys A and S. When key A is down the circuit from the master breaker to the distributor D is closed. When key S is down both this circuit and the set up circuit causing subtraction in all fields during the subsequent cycle of the machine is closed. As indicated by the dotted line these switches are interlocked (probably electrically) with the timing shaft of the tabulator so that the position of the keys (or more precisely the status of the circuits controlled by the keys) can only be altered during the inactive portion of the cycle.

In order to perform the elimination of one variable from two equations we must punch the coefficients and constant term of each equation on cards. To be specific we may imagine we have nine unknowns in the equations and punch these nine coefficients and the constant term in ten fields of eight places thus filling the card. The bar K is moved to its left most, that is the dotted position. The first card is placed in the apparatus and the key A depressed for one cycle. (If A were depressed for two cycles the accuracy of the method would not be impaired). Then the second card is placed in the apparatus and the keys A and S and the position of the bar K are so manipulated that the reading of the counter associated with the variable to be eliminated is reduced to zero. This process is very similar to the process of division on a computing machine in which the remainder carried by the machine is continually reduced. Then it is easy to see mathematically that the numbers carried by the other counters represent the coefficients and constant term of the reduced equation. Upon control the tabulator summary punches these results for a repetition of the process without any chance for human error.

The bar K is to be moved by a mechanical movement that stops it positively in the positions in which the contact springs touch the contacts. In the practical construction of such a device it would probably be advantageous to arrange the contact springs C, C', etc. from a given field, the B springs to supply the "9" and the contacts leading to the corresponding counter in a circular arrangement. The circles for the various fields and counters would be arranged side by side the whole constituting a cylindrical structure. The contact springs could remain stationary and the various circles of contacts would then be made to rotate about a common shaft or vice versa.

In case the counters are not visible (as in the modern alphabet machine) a signal could be arranged to notify the operator of a carry over from the left dial of the counter the contents of which is being reduced to zero. This signal will enable the operator to proceed to reduce the contents of this counter to zero without following its exact contents visually.

5. Automatic Operation

It is comparatively easy to make the elimination of one variable automatic and this is probably desirable since it will reduce operator fatigue and increase the speed of the operation. With this arrangement the operator would add the contents of the first card into the counters, place the second card in the auxiliary device and start the elimination by pressing the A or S key as required to reduce the contents of the counter being eliminated. The machine would then eliminate the contents of this counter, stop and summary punch the results. This effect is obtained in the following way. Suppose the S key is depressed at first to reduce the contents of the counter. The control E will be arranged so that this key will remain depressed until a carry over occurs from the left dial of the counter the contents of which is being eliminated. This closes a set up circuit which remains closed until the machine reaches the succeeding vacant portion of its cycle. Then three things happen automatically. The bar K or its cylindrical equivalent is moved to the next position (i.e. to the right) by an electromagnet, the key S rises and the key A is depressed (this is necessary since we have subtracted once too many times) and the set up circuit is restored. The machine continues adding until carry over then subtracting until carry over with the bar K in succeeding positions until the number in the counter with the carry over signal is eliminated. The last movement of the electromagnet stops the machine and the summary punch punches a card with the coefficients of the new equation. It is estimated that the elimination of one unknown from two equations should be accomplished in less than twenty seconds.

6. More Unknowns

If the number of unknowns is greater than one less than the number of counters on the tabulator an entire equation cannot be punched on one card and the elimination of one unknown cannot be performed in one operation of the tabulator. However the coefficients of a given equation may be punched on two or more cards and the elimination performed in two or more operations if provision is made to punch the coefficients of the variable being eliminated in duplicate on each of the cards representing the equation. The duplicated coefficients can be copied by the summary punch.